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Centers of multilinear forms and applications ☆

Hua-Lin Huang [a], Huajun Lu [a,*], Yu Ye [b], Chi Zhang [c]

[a] *School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China*
[b] *School of Mathematical Sciences, Wu Wen-Tsun Key Laboratory of Mathematics, University of Science and Technology of China, Hefei 230026, China*
[c] *Department of Mathematics, Northeastern University, Shenyang 110819, China*

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ABSTRACT

The center algebra of a general multilinear form is defined and investigated. We show that the center of a nondegenerate multilinear form is a finite dimensional commutative algebra, and center algebras can be effectively applied to direct sum decompositions of multilinear forms. As an application of the algebraic structure of centers, we show that almost all multilinear forms are absolutely indecomposable. The theory of centers can also be applied to symmetric equivalence of multilinear forms. Moreover, with a help of the results of symmetric equivalence, we are able to provide a linear algebraic proof for a well known Torelli type result which says that two complex homogeneous polynomials with the same Jacobian ideal are linearly equivalent.

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* Corresponding author.
E-mail addresses: hualin.huang@hqu.edu.cn (H.-L. Huang), huajun@hqu.edu.cn (H. Lu), yeyu@ustc.edu.cn (Y. Ye), zhangchi@mail.neu.edu.cn (C. Zhang).

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1. Introduction

Let V be an n-dimensional vector space over a field $\Bbbk$. Let $d \geq 3$ be a positive integer. A d-linear form on V is a multilinear mapping $\Theta : V^d = V \times \cdots \times V \to \Bbbk$ and is denoted by (V, Θ) or Θ for short. Take a basis $e_1, e_2, \ldots, e_n$ of V and set $a_{i_1 i_2 \cdots i_d} = \Theta(e_{i_1}, e_{i_2}, \ldots, e_{i_d})$. The resulting tensor $A = \left(a_{i_1 i_2 \cdots i_d}\right)_{1 \leq i_1, i_2, \ldots, i_d \leq n}$ is called the associated tensor of (V, Θ) under the basis $e_1, e_2, \ldots, e_n$. A fundamental problem in invariant theory and multilinear algebra is finding canonical forms for multilinear forms under base change, or equivalently, canonical forms of tensors under congruence by invertible matrices.

Unlike bilinear forms, it seems hopeless to find a complete set of representatives for d-linear forms, see e.g. [3,8]. One of our main concerns is direct sum decompositions of multilinear forms, that is to find whether there exist nonzero subspaces $V_1, V_2, \ldots, V_m$ of (V, Θ) such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ and $\Theta(v_1, \ldots, v_d) = 0$ as long as the v_i's, all taken from $V_1 \cup V_2 \cup \cdots \cup V_m$, are not in the same V_j for some j. In terms of tensors, this is equivalent to finding an invertible matrix P such that the congruent tensor AP^d is block diagonal. This is a natural problem, as direct sum decompositions may provide dimension reduction for multilinear forms.

In [10,11], we studied direct sum decompositions of symmetric multilinear forms via Harrison's theory of centers [9]. The authors showed that the problem can be boiled down to some standard tasks of linear algebra, specifically the computations of eigenvalues and eigenvectors. The main aim of the present paper is to extend [10,11] to the situation of general multilinear forms.

We generalize the key notion of centers as follows.

Definition 1.1. Given a d-linear form (V, Θ), set

$$Z(V,\Theta) \colon = \left\{ \phi \in \mathrm{End}(V) \;\middle|\; \begin{array}{c} \Theta(v_1, \ldots, \phi(v_i), \ldots, v_j, \ldots v_d) \\ = \Theta(v_1, \ldots, v_i, \ldots, \phi(v_j), \ldots, v_d), \\ 1 \leq i, j \leq d, \quad \text{for all } v_1, \ldots, v_d \in V \end{array} \right\} \tag{1.1}$$

and call it the center of (V, Θ).

Elements of centers for multilinear forms were also defined and applied to direct sum decomposition in [4], where they were called self-adjoint linear mappings. However, the algebraic structure of all central, or self-adjoint, elements were not considered therein.

We observe that the centers of multilinear forms enjoy the same properties as those of symmetric multilinear forms, or equivalently homogeneous polynomials, cf. [9–11].

Theorem 1.2. *Suppose (V, Θ) is a nondegenerate d-linear form. Then*

(1) *The center $Z(V, \Theta)$ is a commutative algebra.*

(2) *There is a one-to-one correspondence between direct sum decompositions of* (V, Θ) *and complete sets of orthogonal idempotents of* $Z(V, \Theta)$.
(3) *The decomposition of* (V, Θ) *into a direct sum of indecomposable* d*-linear forms is unique up to permutation of indecomposable summands.*

As a consequence, we have a simple algorithm for direct sum decompositions of arbitrary multilinear forms which is equivalent to the classical eigenvalue problem of matrices, see [10, Algorithm 3.12].

Let $T_{d,n}$ be the set of all d-linear forms on an n-dimensional linear $\Bbbk$-space. If a multilinear form is not a direct sum, then we say it is indecomposable. It is clear by Theorem 1.2 that $(V, \Theta) \in T_{d,n}$ is indecomposable if and only if $Z(V, \Theta)$ is a local algebra. A multilinear form is called absolutely indecomposable, if it remains indecomposable under any extension of the ground field. In particular, if (V, Θ) is central, i.e., $Z(V, \Theta) \cong \Bbbk$, then (V, Θ) is absolutely indecomposable. It was already noticed in [4, Remark 10] that multilinear forms are more likely indecomposable. We confirm this with a help of the center theory. In fact, we show in terms of elementary algebraic geometry that almost all multilinear forms are central, hence are absolutely indecomposable.

Theorem 1.3. *The set of all central* d*-linear forms is Zariski open and dense in* $T_{d,n}$.

We also apply the theory of centers to symmetric equivalence of multilinear forms. This notion was introduced and studied by Belitskii and Sergeichuk in [4]. Let (U, Δ) and (V, Θ) be two d-linear forms. If there exist linear bijections $\phi_1, \ldots, \phi_d : U \to V$ such that

$$\Delta(u_1, \ldots, u_d) = \Theta(\phi_{\sigma_1}(u_1), \ldots, \phi_{\sigma_d}(u_d))$$

for all $u_1, \ldots, u_d \in U$ and each reordering $\sigma_1, \ldots, \sigma_d$ of $1, \ldots, d$, then (U, Δ) and (V, Θ) are called symmetrically equivalent, denoted by $(U, \Delta) \simeq_s (V, \Theta)$. Further, if $\phi_1 = \cdots = \phi_d$, then (U, Δ) and (V, Θ) are called isomorphic, denoted by $(U, \Delta) \cong (V, \Theta)$. Isomorphic multilinear forms are obviously symmetrically equivalent. The converse is not true in general, however, we have

Theorem 1.4. *Let* (U, Δ) *and* (V, Θ) *be two* d*-linear forms.*

(1) *Suppose* $\Delta \simeq_s \Theta$. *Let* $\Delta = \Delta_0 \oplus \Delta_1 \oplus \cdots \oplus \Delta_r$ *(resp.* $\Theta = \Theta_0 \oplus \Theta_1 \oplus \cdots \oplus \Theta_s$*) be the decomposition of* Δ *(resp.* Θ*) as the direct sum of a zero form and indecomposable* d*-linear forms where* Δ_0 *and* Θ_0 *are zero forms and the other* Δ_i*'s and* Θ_i*'s are indecomposable. Then we have* $r = s$ *and* $\Delta_i \simeq_s \Theta_i$ *for each* i *after suitable reordering of* Θ_i*'s.*
(2) *Suppose the characteristic of* $\Bbbk$ *is zero or coprime to* d. *Assume further that* Δ *and* Θ *are absolutely indecomposable. Then* $\Delta \simeq_s \Theta$ *if and only if* $\Delta \cong a\Theta$ *for some nonzero* $a \in \Bbbk$.

(3) *Suppose* $\Bbbk$ *is algebraically closed and its characteristic is zero or coprime to* d. *Then* $\Delta \simeq_s \Theta$ *if and only if* $\Delta \cong \Theta$.

This generalizes the related results of [4], and the arguments are considerably simplified with the help of centers. Moreover, one can define centers of multilinear maps [2] and obtain results similar to Theorems 1.2, 1.3 and 1.4. Interestingly enough, the previous results of symmetric equivalence of multilinear forms can be applied to provide a simple linear algebraic proof for a well known Torelli type result of Donagi [6, Proposition 1.1]. To the best of our knowledge, the previously known proofs are more or less analytic and sophisticated.

Theorem 1.5. *Suppose the field* $\Bbbk$ *is algebraically closed and its characteristic is* 0 *or greater than* d. *If* f *and* g *are two homogeneous polynomials of degree* d *with the same Jacobian ideal, then they are related by an invertible linear transformation.*

Throughout, we assume that d is an integer greater than 2, $\Bbbk$ is a field of characteristic 0 or greater than d, unless otherwise stated. The results are presented in terms of multilinear forms. We leave the equivalent version for tensors to the interested reader. Theorems 1.2 and 1.3 are proved in Section 2, Theorem 1.4 is proved in Section 3, and Theorem 1.5 is proved in Section 4.

2. Centers and direct sum decompositions of multilinear forms

In this section, we consider the center algebras of multilinear forms with applications to direct sum decompositions. First of all, we recall some concepts.

Definition 2.1. Let (V, Θ) be a d-linear form. If there exist nonzero subspaces $V_1, \dots, V_s$ ($s \geq 2$) of (V, Θ) such that $V = V_1 \oplus \cdots \oplus V_s$ and $\Theta(v_1, \dots, v_d) = 0$ for all $v_1, \dots, v_d \in \bigcup_{i=1}^{s} V_i$ unless all the v_i's are in the same V_k for some k, then Θ is called the (inner) direct sum of $\Theta_1, \dots, \Theta_s$, where $\Theta_i = \Theta|_{V_i}$ is the restriction of Θ to V_i for $1 \leq i \leq s$ and we denote it by $(V, \Theta) = (V_1, \Theta_1) \oplus \cdots \oplus (V_s, \Theta_s)$. We call (V, Θ) decomposable if it is a direct sum. Otherwise, we call (V, Θ) indecomposable.

Similar to the symmetric case [9], there is no harm in assuming that the d-linear form (V, Θ) is nondegenerate, that is, $u = 0$ is the only solution to the following linear equations

$$\Theta(u, v_1, \dots, v_{d-1}) = \Theta(v_1, u, \dots, v_{d-1}) = \cdots = \Theta(v_1, \dots, v_{d-1}, u) = 0 \tag{2.1}$$

for all $v_1, \dots, v_{d-1} \in V$. For an arbitrary d-linear form (V, Θ), let V_0 be the solution space of the previous equations (2.1) and take a subspace V_1 of V such that $V = V_0 \oplus V_1$, then $(V, \Theta) = (V_0, \Theta_0) \oplus (V_1, \Theta_1)$. It is immediate that a degenerate d-linear form is decomposable. In particular, (V_0, Θ_0) is a zero form and (V_1, Θ_1) is nondegenerate. Note moreover that V_0 is uniquely determined by Θ, and (V_1, Θ_1) is uniquely determined by Θ up to isomorphism, see also [4, Theorem 9].

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. (1) Let us show that $Z(V, \Theta)$ is a commutative subalgebra of $\mathrm{End}(V)$. It is obvious that $Z(V, \Theta)$ is closed under linear combinations. Choose two arbitrary $\phi, \psi \in Z(V, \Theta)$ and we want to show $\phi \circ \psi \in Z(V, \Theta)$. According to the definition of centers, for all $v_1, \ldots, v_d \in V$ we have

$$\begin{aligned}\Theta(v_1, \ldots, \phi \circ \psi(v_i), \ldots, v_j, \ldots v_d) &= \Theta(v_1, \ldots, \psi(v_i), \ldots, v_j, \ldots, \phi(v_d)) \\ &= \Theta(v_1, \ldots, v_i, \ldots, \psi(v_j), \ldots, \phi(v_d)) \\ &= \Theta(v_1, \ldots, v_i, \ldots, \phi \circ \psi(v_j), \ldots, v_d).\end{aligned}$$

Hence we have $\phi \circ \psi \in Z(V, \Theta)$. Similarly we show the commutativity of $Z(V, \Theta)$ as follows.

$$\begin{aligned}\Theta(v_1, \ldots, \phi \circ \psi(v_i), \ldots, v_j, \ldots v_d) &= \Theta(v_1, \ldots, \psi(v_i), \ldots, \phi(v_j), \ldots v_d) \\ &= \Theta(v_1, \ldots, v_i, \ldots, \phi(v_j), \ldots, \psi(v_d)) \\ &= \Theta(v_1, \ldots, \phi(v_i), \ldots, v_j, \ldots, \psi(v_d)) \\ &= \Theta(v_1, \ldots, \psi \circ \phi(v_i), \ldots, v_j, \ldots v_d).\end{aligned}$$

We conclude that $\Theta(v_1, \ldots, [\phi \circ \psi - \psi \circ \phi](v_i), \ldots, v_j, \ldots v_d) = 0$ for all $v_1, \ldots, v_d \in V$. As (V, Θ) is nondegenerate, it follows that $\phi \circ \psi - \psi \circ \phi = 0$, that is, $\phi \circ \psi = \psi \circ \phi$.

(2) Suppose $(V, \Theta) = (V_1, \Theta_1) \oplus \cdots \oplus (V_s, \Theta_s)$ is a direct sum decomposition. For $1 \le i \le s$, let $e_i : V \twoheadrightarrow V_i \hookrightarrow V$ be the composition of the canonical projection $V \twoheadrightarrow V_i$ and the embedding $V_i \hookrightarrow V$. Then it is obvious that $e_i^2 = e_i,\ e_i e_j = 0$ whenever $i \neq j$, and by definition it is easy to verify that each $e_i \in Z(V, \Theta)$. In other words, $e_1, \ldots, e_s$ are a complete set of orthogonal idempotents of $Z(V, \Theta)$.

Conversely, suppose $e_1, \ldots, e_s$ are a complete set of orthogonal idempotents of $Z(V, \Theta)$. Let $V_i = e_i V$ and $\Theta_i = \Theta|_{V_i}$. Then it is not hard to verify that $(V_1, \Theta_1) \oplus \cdots \oplus (V_s, \Theta_s)$ is a direct sum decomposition of (V, Θ). Indeed, assume $v_1, \ldots, v_d$ are taken from the subspaces V_i's and $v_j \in V_j,\ v_k \in V_k$ with $j < k$, then

$$\begin{aligned}&\Theta(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_d) \\ &= \Theta(v_1, \ldots, e_j v_j, \ldots, e_k v_k, \ldots, v_d) \\ &= \Theta(v_1, \ldots, v_j, \ldots, e_j e_k v_k, \ldots, v_d) \\ &= 0.\end{aligned}$$

(3) It suffices to prove that $Z(V, \Theta)$ has a unique complete set of primitive orthogonal idempotents disregarding their order thanks to (2). Suppose $1 = e_1 + \cdots + e_s = f_1 + \cdots + f_t$

where all e_i's and f_j's are primitive orthogonal idempotents. Then for any fixed i, $e_i = e_i(f_1+\cdots+f_t) = e_if_1+\cdots+e_if_t$. Since $(e_if_j)^2 = e_i^2f_j^2 = e_if_j$ and e_i is primitive, $e_i = e_if_j$ for some certain j. Similarly, $f_j = f_je_k$ for some certain k. We claim that $i = k$, and thus $e_i = e_if_j = f_je_i = f_j$. Otherwise, if $i \neq k$, then $e_i = e_if_j = e_if_je_k = e_ie_kf_j = 0$. This is absurd. Then we are done. □

Remarks 2.2. Keep the assumption that (V, Θ) is a nondegenerate d-linear form.

(1) (V, Θ) is indecomposable if and only if $Z(V, \Theta)$ is a local algebra.
(2) The uniqueness of direct sum decomposition of multilinear forms were dealt with by other approaches in [9, Proposition 2.3] (the symmetric case) and [4, Theorem 9]. The treatment via centers seems much more convenient.
(3) The algorithm of direct sum decomposition of symmetric multilinear forms proposed by the authors [11, Algorithm 3.12] can be extended verbatim to the present situation.

Now we give some examples of the centers of multilinear forms. First of all, it is convenient to turn (1.1) in the definition of centers into explicit linear equations. Assume that V is an n-dimensional $\Bbbk$-space with a basis $e_1, \ldots, e_n$. Let $A = \big(a_{i_1i_2\cdots i_d}\big)_{1\le i_1,i_2,\ldots,i_d\le n}$ be the associated tensor of (V, Θ) under the basis $e_1, e_2, \ldots, e_n$. Then we have

$$Z(V,\Theta) \cong \{X \in \Bbbk^{n\times n} \mid X^T A_{i_1\cdots \underline{i_k}\cdots \underline{i_l}\cdots i_d} = A_{i_1\cdots \underline{i_k}\cdots \underline{i_l}\cdots i_d}X,\ \ 1 \le i_1,\ldots,i_d \le n\}, \quad (2.2)$$

where $A_{i_1\cdots \underline{i_k}\cdots \underline{i_l}\cdots i_d}$ denotes the $n \times n$ matrix $\big(a_{i_1\cdots i_{k-1},i,i_{k+1}\cdots i_{l-1},j,i_{l+1}\cdots i_d}\big)_{1\le i,j\le n}$.

Example 2.3. Let V be the 3-dimensional Euclidean space and consider the scalar triple product. Given arbitrary three vectors $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$, we define a 3-linear form $\Theta(x, y, z) = x_1y_1z_1 + x_2y_2z_2 + x_3y_3z_3$. Let $A = \big(a_{ijk}\big)_{1\le i,j,k\le 3}$ be the associated tensor of (V, Θ) under the canonical basis e_1, e_2, e_3 of $\mathbb{R}^3$. Then $a_{iii} = 1$ for all i and $a_{ijk} = 0$ if i, j, k are not identical. Suppose $X = \big(x_{ij}\big)_{3\times 3} \in Z(V, \Theta)$. Note that A is symmetric, that is invariant under permutation of indices, so it is enough to consider the equations $X^T A_{i\underline{i_2}\underline{i_3}} = A_{i\underline{i_2}\underline{i_3}}X$ for $i = 1, 2, 3$ by (2.2). As $A_{i\underline{i_2}\underline{i_3}}$ has all 0 entries but (i, i)-entry 1, by easy computations one has $x_{ij} = 0$ whenever $j \neq i$. Hence the center $Z(V, \Theta)$ consists of all the diagonal matrices and we have $Z(V, \Theta) \cong \mathbb{R}^3$. It follows by Theorem 1.2 that (V, Θ) is a direct sum of 3 one-dimensional 3-linear forms. Indeed, let V_i be the space spanned by e_i and Θ_i the restriction of Θ to V_i, then it is clear that $(V, \Theta) = (V_1, \Theta_1) \oplus (V_2, \Theta_2) \oplus (V_3, \Theta_3)$.

Example 2.4. Consider the space $V = \Bbbk^n$ of all the n-dimensional column vectors. Given arbitrary n vectors $v_1, \ldots, v_n$, we define an n-linear form $\Theta(v_1, \ldots, v_n) = \det M$, where M is the $n \times n$ matrix with columns $v_1, \ldots, v_n$. The associated tensor $A = \big(a_{i_1\cdots i_n}\big)_{1\le i_1,\ldots,i_n\le n}$ of Θ with respect to the canonical basis of V satisfies $a_{i_1\cdots i_n} = 0$

unless $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$, in which case $a_{i_1 \cdots i_n}$ is the sign of the permutation $\begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix}$. For a permutation $i_1, \ldots, i_n$, suppose $i_k = i$ and $i_l = j$. Then the matrix $A_{i_1 \cdots \underline{i_k} \cdots \underline{i_l} \cdots i_n}$ is anti-symmetric and has only two nonzero entries with the value ± 1 at (i,j)-entry and ∓ 1 at (j,i)-entry. Let $X = \left(x_{kl}\right)_{n \times n} \in Z(V, \Theta)$ and solve the matrix equation $X^T A_{i_1 \cdots \underline{i_k} \cdots \underline{i_l} \cdots i_n} = A_{i_1 \cdots \underline{i_k} \cdots \underline{i_l} \cdots i_n} X$. One has easily $x_{ii} = x_{jj}$, and $x_{is} = x_{jt} = 0$ whenever $s \neq i, t \neq j$. Running over all the possible pairs (i,j)'s, we show that the center $Z(V, \Theta)$ consists of all the scalar matrices and thus $Z(V, \Theta) \cong \Bbbk$. Therefore, the multilinear form (V, Θ) is indecomposable by Remarks 2.2.

Example 2.5. Let V be the algebra of $n \times n$-matrices. For any d matrices $M_1, \ldots, M_d$, we define $\Theta(M_1, \ldots, M_d) = \operatorname{tr}(M_1 \cdots M_d)$ where tr is the trace map of matrices. Let $E_{ij} \in V$ be the matrix unit which has a 1 in the (i,j) position as its only nonzero entry. Let $A = \left(a_{i_1 j_1 \cdots i_d j_d}\right)_{1 \leq i_1, \ldots, i_d, j_1, \ldots, j_d \leq n}$ be the associated tensor of Θ under the basis $\{E_{ij}, 1 \leq i,j \leq n\}$. Then we have $a_{i_1 j_1 \cdots i_d j_d} = 1$ when $j_d = i_1, j_k = i_{k+1}, 1 \leq k \leq d-1$ and $a_{i_1 j_1 \cdots i_d j_d} = 0$ otherwise. Similar to the previous example, by direct computation one can show that the center $Z(V, \Theta) \cong \Bbbk$.

Example 2.6. Take any associative algebra A with unit and any linear function ϕ on A. Consider the d-linear form $\Theta(a_1, \ldots, a_d) = \phi(a_1 \cdots a_d)$. Let Z be the usual center of A. For each $a \in A$, let l_a be the endomorphism of A sending each $x \in A$ to ax. If $a \in Z$, then it is easy to see that $l_a \in Z(\Theta)$. Therefore we have an embedding from Z into $Z(A, \Theta)$. In the previous example, these two centers are isomorphic to each other. However, in general the embedding is not surjective. For example, let $A = \Bbbk[t]/(t^2)$ and $\phi(x + y\bar{t}) = x$ for all $x, y \in \Bbbk$, where $\bar{t}$ denotes the congruence class of t. Let Ψ be the linear endomorphism of A such that $\Psi(x + y\bar{t}) = y\bar{t}$. As $\phi(\Psi(x + y\bar{t})(z + w\bar{t})) = 0$ for all $w, x, y, z \in \Bbbk$, we conclude that $\Psi \in Z(A, \Theta)$. However Ψ is not any l_a with $a \in A$. This means that the center of multilinear forms is a nontrivial extension of the usual center of algebras.

In the rest of this section, we consider the algebraic structure of the center of a general d-linear form. This may provide important structural information for d-linear forms. Our chief concern is whether a general d-linear form is decomposable. It was already noticed in [4, Remark 10] that multilinear forms are more likely indecomposable. This is confirmed in terms of elementary algebraic geometry with a help of the center theory. We will show that almost all multilinear forms have trivial center, namely the center is isomorphic to the ground field. Such multilinear forms are called central. Clearly, a central multilinear form is indecomposable by item (1) of Remarks 2.2. The relevant result for symmetric multilinear forms was proved in [11, Theorem 3.2], where the same idea can be extended to the present situation.

First we construct some examples of central multilinear forms in general degree and dimension. This is necessary for the argument in the proof of Theorem 1.3.

Example 2.7. We construct a d-linear form with trivial center for each $d \geq 3$ and $n \geq 2$.

If $n = 2$, let $(a_{ij})_{2\times 2} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Let Θ be the d-linear form such that $a_{i_1 i_2 i_3 \cdots i_d} = a_{i_1 i_2}$ for all $1 \leq i_1, \ldots, i_d \leq 2$. An easy calculation shows that $Z(V, \Theta) \cong \Bbbk$.

If $n \geq 3$, let $p_1, \ldots, p_n, q_1 \ldots, q_n$ be $2n$ nonzero elements of $\Bbbk$ such that $\frac{p_j}{p_i} \neq \frac{q_j}{q_i}$ whenever $i \neq j$. Let $A_1 = \big(a_{ij}^{(1)}\big)$ (resp. $A_2 = \big(a_{ij}^{(2)}\big)$) be the diagonal $n \times n$ matrix with $a_{ii}^{(1)} = p_i$ (resp. $a_{ii}^{(2)} = q_i$) for $1 \leq i \leq n$. Let $A_3 = (a_{ij}^{(3)})$ be the matrix with $a_{ij}^{(3)} = 1$ for $1 \leq i, j \leq n$. Let Θ be the d-linear form such that $a_{1 i_2 i_3 \cdots i_d} = a_{i_2 i_3}^{(1)}$, $a_{2 i_2 i_3 \cdots i_d} = a_{i_2 i_3}^{(2)}$, $a_{3 i_2 i_3 \cdots i_d} = a_{i_2 i_3}^{(3)}$, $a_{i i_2 i_3 \cdots i_d} = 0$ for $4 \leq i \leq n, 1 \leq i_2, i_3, \ldots, i_d \leq n$. Suppose $X = \big(x_{ij}\big)_{n\times n} \in Z(V, \Theta)$, then we have $X^T A_i = A_i X$ for $i = 1, 2, 3$. Consequently, we have $p_i x_{ij} = p_j x_{ji}$ and $q_i x_{ij} = q_j x_{ji}$ for all $1 \leq i, j \leq n$. As $\frac{p_j}{p_i} \neq \frac{q_j}{q_i}$ whenever $i \neq j$, we conclude that X must be a diagonal matrix. Moreover, X must be a multiple of the identity matrix since $X^T A_3 = A_3 X$. Therefore, we have $Z(V, \Theta) \cong \Bbbk$.

Proof of Theorem 1.3. Let C be the set of all central d-linear forms. Clearly, C is not empty by Example 2.7. See also [11] for many other examples of central d-linear forms which are symmetric. The center of (V, Θ) is the solution space to a system of linear equations on x_{ij}'s: $X^T A_{i_1 \cdots \underline{i_k} \cdots \underline{i_l} \cdots i_d} = A_{i_1 \cdots \underline{i_k} \cdots \underline{i_l} \cdots i_d} X$ for all possible index $i_1 \cdots i_d$, where we use the same notations as in Equation (2.2). The d-linear form (V, Θ) is central if and only if the rank of the coefficient matrix, denoted by B, of the linear system (2.2) is equal to $n^2 - 1$. Hence C is the union of all the principal open sets defined by the $(n^2 - 1)$-minors of B regarding all $a_{i_1 \cdots i_d}$'s as indeterminates. Consequently, C is a nonempty Zariski open set of $T_{d,n}$ and so is dense. □

3. Symmetric equivalence of multilinear forms

This section is motivated by [4]. We apply the theory of centers to investigate symmetric equivalence of multilinear forms. First, we recall some notions.

Definition 3.1. Let (U, Δ) and (V, Θ) be two d-linear forms.

(1) (U, Δ) and (V, Θ) are called symmetrically equivalent, denoted by $(U, \Delta) \simeq_s (V, \Theta)$, if there exist linear bijections $\phi_1, \ldots, \phi_d : U \to V$ such that

$$\Delta(u_1, \ldots, u_d) = \Theta(\phi_{\sigma_1}(u_1), \ldots, \phi_{\sigma_d}(u_d))$$

for all $u_1, \ldots, u_d \in U$ and each reordering $\sigma_1, \ldots, \sigma_d$ of $1, \ldots, d$.

(2) (U, Δ) and (V, Θ) are called isomorphic, denoted by $(U, \Delta) \cong (V, \Theta)$, if there exists a linear bijection $\phi : U \to V$ such that

$$\Delta(u_1, \ldots, u_d) = \Theta(\phi(u_1), \ldots, \phi(u_d))$$

for all $u_1, \ldots, u_d \in U$.

(3) The (outer) direct sum of (U, Δ) and (V, Θ) is the d-linear form $\Delta \oplus \Theta : (U \oplus V)^d \to \Bbbk$ defined by

$$(\Delta \oplus \Theta)(u_1 + v_1, \ldots, u_d + v_d) = \Delta(u_1, \ldots, u_d) + \Theta(v_1, \ldots, v_d)$$

for all $u_1, \ldots, u_d \in U$ and $v_1, \ldots, v_d \in V$.

Remark 3.2. Isomorphic forms are called congruent in [4]. We leave the terminology congruence for the associated tensors of multilinear forms in consideration. If two forms are isomorphic, then evidently they are symmetrically equivalent. The converse is not true in general. For example, let (V, Θ) be the $2d$-linear form over the field $\mathbb{R}$ of real numbers defined by $\Theta(e_i, e_i, \ldots, e_i) = 1,\ 1 \le i \le n$, and $\Theta(e_{i_1}, e_{i_2}, \ldots, e_{i_{2d}}) = 0$ otherwise. It is clear that (V, Θ) is not isomorphic to $(V, -\Theta)$, as the former is positive definite, while the latter is negative definite. However, it is easy to see that the bijections $-\mathrm{Id}, \mathrm{Id}, \ldots, \mathrm{Id}$ make a symmetric equivalence between them.

Now we investigate symmetric equivalence of d-linear forms via their centers.

Proposition 3.3. *Suppose (U, Δ) and (V, Θ) are nondegenerate d-linear forms. If $(U, \Delta) \simeq_s (V, \Theta)$, then $Z(U, \Delta)$ is isomorphic to $Z(V, \Theta)$ as algebras.*

Proof. Let $\phi_1, \ldots, \phi_d : U \to V$ be the linear bijections such that

$$\Delta(u_1, \ldots, u_d) = \Theta(\phi_{\sigma_1}(u_1), \ldots, \phi_{\sigma_d}(u_d))$$

for all $u_1, \ldots, u_d \in U$ and each reordering $\sigma_1, \ldots, \sigma_d$ of $1, \ldots, d$. In particular, for fixed bijections ϕ_k and ϕ_l we have

$$\begin{aligned}
&\Delta(u_1, \ldots, \phi_k^{-1}\phi_l(u_i), \ldots, u_j, \ldots, u_d) \\
&= \ \Theta(\phi_{\sigma_1}(u_1), \ldots, \phi_k(\phi_k^{-1}\phi_l(u_i)), \ldots, \phi_l(u_j), \ldots,\ \phi_{\sigma_d}(u_d)) \ \ (\phi_{\sigma_i} = \phi_k, \phi_{\sigma_j} = \phi_l) \\
&= \Theta(\phi_{\sigma_1}(u_1), \ldots, \phi_l(u_i), \ldots, \phi_l(u_j), \ldots, \phi_{\sigma_d}(u_d)) \\
&= \Theta(\phi_{\sigma_1}(u_1), \ldots, \phi_l(u_i), \ldots, \phi_k(\phi_k^{-1}\phi_l(u_j)), \ldots, \phi_{\sigma_d}(u_d)) \\
&= \Delta(u_1, \ldots, u_i, \ldots, \phi_k^{-1}\phi_l(u_j), \ldots, u_d).
\end{aligned}$$

Therefore we show that $\phi_k^{-1}\phi_l \in Z(U, \Delta)$ for all possible pairs (k, l). Similarly, we can show that $\phi_k\phi_l^{-1} \in Z(V, \Theta)$ for all possible pairs (k, l).

As $Z(V, \Theta)$ is commutative, for any $\phi \in Z(U, \Delta)$ we have

$$\phi_k\phi\phi_k^{-1} \circ (\phi_l\phi\phi_l^{-1})^{-1} = \phi_k\phi(\phi_k^{-1}\phi_l)\phi^{-1}\phi_l^{-1} = \phi_k\phi\phi^{-1}(\phi_k^{-1}\phi_l)\phi_l^{-1} = \mathrm{Id}_V,$$

where Id_V is the identity map on V. Therefore, $\phi_k\phi\phi_k^{-1} = \phi_l\phi\phi_l^{-1}$ holds for all possible pairs (k, l).

Since

$$\begin{aligned}
&\Theta(v_1, \dots, \phi_k\phi\phi_k^{-1}(v_i), \dots, v_j, \dots, v_d)\\
&= \Delta(\phi_{\sigma_1}^{-1}(v_1), \dots, \phi_k^{-1}\phi_k\phi\phi_k^{-1}(v_i), \dots, \phi_l^{-1}(v_j), \dots, \phi_{\sigma_d}^{-1}(v_d)) \quad (\phi_{\sigma_i}^{-1} = \phi_k^{-1}, \phi_{\sigma_j}^{-1} = \phi_l^{-1})\\
&= \Delta(\phi_{\sigma_1}^{-1}(v_1), \dots, \phi\phi_k^{-1}(v_i), \dots, \phi_l^{-1}(v_j), \dots, \phi_{\sigma_d}^{-1}(v_d))\\
&= \Delta(\phi_{\sigma_1}^{-1}(v_1), \dots, \phi_k^{-1}(v_i), \dots, \phi\phi_l^{-1}(v_j), \dots, \phi_{\sigma_d}^{-1}(v_d))\\
&= \Theta(v_1, \dots, v_i, \dots, \phi_l\phi\phi_l^{-1}(v_j), \dots, v_d)\\
&= \Theta(v_1, \dots, v_i, \dots, \phi_k\phi\phi_k^{-1}(v_j), \dots, v_d),
\end{aligned}$$

we conclude $\phi_k\phi\phi_k^{-1} \in Z(V, \Theta)$ for all $1 \le k \le d$. Finally, we can construct the isomorphism $\Psi : Z(U, \Delta) \to Z(V, \Theta)$ by $\Psi(\phi) = \phi_1\phi\phi_1^{-1}$. □

Before proving the main results on symmetric equivalence of multilinear forms, we need some technical preparations particularly in commutative algebra, and see e.g. [1,7]. A d-linear form (V, Θ) over $\Bbbk$ is called absolutely indecomposable if it remains indecomposable after any field extension of $\Bbbk$. A central from is obviously absolutely indecomposable.

Lemma 3.4. *Suppose the characteristic of $\Bbbk$ is zero or coprime to d. Let A be a commutative finite dimensional local $\Bbbk$-algebra with maximal ideal m. Let $K = A/m$ be its residue field. Then we have $A^\times/(A^\times)^d \cong K^\times/(K^\times)^d$, where $A^\times$ (resp. $K^\times$) is the group of units of A (resp. K). Moreover, if $K/\Bbbk$ is purely inseparable, then $K^\times/(K^\times)^d \cong \Bbbk^\times/(\Bbbk^\times)^d$.*

Proof. As A is local, we have an exact sequence $1 \longrightarrow 1+m \longrightarrow A^\times \longrightarrow K^\times \longrightarrow 1$. After tensoring with $\mathbb{Z}/d\mathbb{Z}$, we obtain the following exact sequence

$$1+m/(1+m)^d \longrightarrow A^\times/(A^\times)^d \longrightarrow K^\times/(K^\times)^d \longrightarrow 1 .$$

Since the characteristic of $\Bbbk$ is zero or coprime to d, for each $a \in 1+m$ the equation $X^d - a = 0$ always has a solution in A by Hensel's Lemma [7, Theorem 7.3]. Therefore each element of $1+m$ is a d-th power, and we have $A^\times/(A^\times)^d \cong (K)^\times/(K^\times)^d$.

Suppose $K/\Bbbk$ is purely inseparable. Then either $K = \Bbbk$, or the characteristic of $\Bbbk$ is a prime, see e.g. [12, §6 of Chap. V]. Obviously, it suffices to consider the latter case. The canonical morphism $\Bbbk \to A \to K$ induces the map $\phi : \Bbbk^\times/(\Bbbk^\times)^d \to K^\times/(K^\times)^d$. First, we show that ϕ is surjective. Let $p = \mathrm{char}\Bbbk$, then for each $a \in K$, there exists certain p^n such that $b = a^{p^n} \in \Bbbk$ as $K/\Bbbk$ is purely separable. As $(p, d) = 1$, there exist two integers α, β such that $p^n\alpha + d\beta = 1$. Therefore, $a = a^{p^n\alpha+d\beta} = b^\alpha a^{d\beta} \in \Bbbk^\times(K^\times)^d$ and ϕ is surjective. On the other side, if there exists an element $c \in \Bbbk$ such that $c = u^d$ for some

$u \in K$, then u is a root of the separable polynomial $T^d - c$ and $\Bbbk(u)/\Bbbk$ is a separable subextension of $K/\Bbbk$. However, as $K/\Bbbk$ is purely inseparable, this forces $\Bbbk(u) = \Bbbk$, that is, $u \in \Bbbk$. Thus, we have shown that ϕ is an isomorphism and the proof is completed. □

Proposition 3.5. *A d-linear form (V, Θ) is absolutely indecomposable if and only if $Z(V, \Theta)$ is local and its residue field is purely inseparable over $\Bbbk$.*

Proof. A form Θ is absolutely indecomposable if and only if $Z(V, \Theta) \otimes k'$ is local for any field extension $k'/\Bbbk$ by Remark 2.2. Let m be the maximal ideal of $Z(V, \Theta)$ and $K = Z(V, \Theta)/m$. As m is nilpotent, $Z(V, \Theta) \otimes k'$ is local if and only if $K \otimes k'$ is local. Therefore, it is enough to show that $K/\Bbbk$ is purely inseparable if and only if $K \otimes k'$ is local for any field extension $k'/\Bbbk$.

If $K/\Bbbk$ is purely inseparable, then $K = \Bbbk$ and the claim is obvious, or $p = \mathrm{char}(\Bbbk) > 0$ and some p power of any element of $K \otimes k'$ belongs to $\Bbbk \otimes k' = k'$. In the latter case, the element of $K \otimes k'$ is either nilpotent or invertible, consequently $K \otimes k'$ is local. On the other side, if $K/\Bbbk$ is not purely inseparable, then there exists a maximal separable subextension $M/\Bbbk$ of degree $r > 1$. Let $\Bbbk^{alg}$ be the algebraic closure of $\Bbbk$, and we have $K \otimes \Bbbk^{alg} \cong K \otimes_M (M \otimes_{\Bbbk} \Bbbk^{alg}) \cong (K \otimes_M \Bbbk^{alg})^r$, which clearly is not local. Therefore we finish the proof. □

Proof of Theorem 1.4. (1) Let $U_0 = \{u \in U \mid \Theta(u, v_1, \ldots, v_{d-1}) = \Theta(v_1, u, \ldots, v_{d-1}) = \cdots = \Theta(v_1, \ldots, v_{d-1}, u) = 0$, for all $v_1, \ldots, v_{d-1} \in U\}$. Let $\overline{U}$ be the quotient space U/U_0 and define the induced d-linear form $\overline{\Delta} : \overline{U} \times \cdots \times \overline{U} \to \Bbbk$ by $\overline{\Delta}(\overline{u_1}, \ldots, \overline{u_n}) = \Delta(u_1, \ldots, u_d)$, where $u_i \in U$ is a lifting of $\overline{u_i}$ for each $1 \le i \le n$. Then $\overline{\Delta}$ is nondegenerate by construction. Moreover, it is easy to verify that $(U, \Delta) \cong (U_0, 0) \oplus (\overline{U}, \overline{\Delta})$. Similarly, we construct $V_0, (\overline{V} = V/V_0, \overline{\Theta})$ and have $(V, \Theta) \cong (V_0, 0) \oplus (\overline{V}, \overline{\Theta})$.

Let $\phi_1, \ldots, \phi_d : U \to V$ be the linear bijections that define $(U, \Delta) \simeq_s (V, \Theta)$. Then $\phi_i(U_0) \subset V_0$ and $\phi_i^{-1}(V_0) \subset U_0$ for each $1 \le i \le d$ by their definitions. Thus ϕ_i induces linear bijections between U_0 (resp. $\overline{U}$) and V_0 (resp. $\overline{V}$) for each $1 \le i \le d$. Clearly, $(U_0, 0)$ and $(V_0, 0)$ are isomorphic as d-linear forms. Let $\overline{\phi_i}$ denote the induced bijection between $\overline{U}$ and $\overline{V}$, then $\overline{\phi_1}, \ldots, \overline{\phi_d}$ give a symmetric equivalence between $\overline{\Delta}$ and $\overline{\Theta}$. Therefore, we can reduce the proposition to the nondegenerate case, that is, we may assume Δ and Θ are nondegenerate in the following.

By Theorem 1.2, we can uniquely decompose $(U, \Delta) = (U_1, \Delta_1) \oplus \cdots \oplus (U_r, \Delta_r)$ as an inner direct sum of indecomposable summands such that each Δ_i is associated to a primitive idempotent e_i of the center $Z(U, \Delta)$ and $U_i = e_i(U)$. As $\Delta \simeq_s \Theta$, we have $Z(U, \Delta) \cong Z(V, \Theta)$ via sending each $\phi \in Z(U, \Delta)$ to $\phi_1 \phi \phi_1^{-1}$ by Proposition 3.3. Again by Theorem 1.2, we obtain the unique direct sum decomposition $(V, \Theta) = (V_1, \Theta_1) \oplus \cdots \oplus (V_r, \Theta_r)$ where $V_i = \phi_1 e_i \phi_1^{-1}(V)$ for each i. In addition, the restrictions of ϕ_k's on U_i give a symmetric equivalence between Δ_i and Θ_i for each i, since $\phi_k(U_i) = \phi_k e_i(U) = \phi_k e_i \phi_k^{-1} \phi_k(U) = \phi_1 e_i \phi_1^{-1}(V) = V_i$ by the proof of Proposition 3.3.

(2) If $\Delta \cong \Theta$, it is obvious that $\Delta \simeq_s \Theta$. Conversely, if $\Delta \simeq_s \Theta$, then $\phi_i\phi_j^{-1} \in Z(V,\Theta)$ for all $1 \leq i, j \leq d$ according to the proof of Proposition 3.3. For each i, we can write $\phi_i = a_i\phi_1$ for some $a_i \in Z(V,\Theta)^\times$. As (V,Θ) is absolutely indecomposable, its center $Z(V,\Theta)$ is a commutative finite dimensional local algebra with purely inseparable residue field by Proposition 3.5. By Lemma 3.4, there exists an $a \in \Bbbk^\times$ such that the product $a^{-1} \cdot a_1 \cdots a_d$ has a d-th root $b \in Z(V,\Theta)^\times$. Since

$$\begin{aligned}
\Delta(u_1,\ldots,u_d) &= \Theta(\phi_1(u_1),\ldots,\phi_d(u_d)) \\
&= \Theta(a_1\phi_1(u_1),\ldots,a_d\phi_1(u_d)) \\
&= \Theta(a_1\cdots a_d\phi_1(u_1),\phi_1(u_2),\ldots,\phi_1(u_d)) \\
&= a\Theta(a^{-1}\cdot a_1\cdots a_d\phi_1(u_1),\phi_1(u_2),\ldots,\phi_1(u_d)) \\
&= a\Theta(b^d\phi_1(u_1),\phi_1(u_2),\ldots,\phi_1(u_d)) \\
&= a\Theta(b\phi_1(u_1),b\phi_1(u_2),\ldots,b\phi_1(u_d)),
\end{aligned}$$

we have $\Delta \cong a\Theta$.

(3) Suppose $(U,\Delta) \simeq_s (V,\Theta)$. By item (1), we have $(U,\Delta) \cong (U_0,0) \oplus (U_1,\Delta_1) \oplus \cdots \oplus (U_r,\Delta_r)$ and $(V,\Theta) \cong (V_0,0) \oplus (V_1,\Theta_1) \oplus \cdots \oplus (V_r,\Theta_r)$ such that all Δ_i's and Θ_i's are indecomposable and $\Delta_i \simeq_s \Theta_i$ for each i. As $\Bbbk$ is algebraically closed, Δ_i and Θ_i are absolutely indecomposable. Then for each i, we have $\Delta_i \cong a_i\Theta_i$ for some $a_i \in \Bbbk^\times$ by item (2). Clearly $a_i\Theta_i \cong \Theta_i$ as a_i has d-th roots in $\Bbbk$, and consequently we have proved $\Delta \cong \Theta$. The converse is trivial and we finish the proof. □

As direct consequences, we recover easily the related main results of [4, Theorem 2].

Corollary 3.6.

(1) *Two d-linear forms over the complex field $\mathbb{C}$ are isomorphic if and only if they are symmetrically equivalent.*
(2) *When d is a positive odd integer, two d-linear forms over the real field $\mathbb{R}$ are isomorphic if and only if they are symmetrically equivalent.*
(3) *Assume d is a positive even integer. If two d-linear forms (U,Δ) and (V,Θ) over the real field $\mathbb{R}$ are symmetrically equivalent, then there exist direct sum decompositions $\Delta = \Delta_1 \oplus \Delta_2$ and $\Theta = \Theta_1 \oplus \Theta_2$ such that $\Delta_1 \cong \Theta_1$ and $\Delta_2 \cong -\Theta_2$.*

Proof. Items (1) and (2) are straightforward by (2) and (3) of Theorem 1.4. For item (3), first we assume further that (U,Δ) and (V,Θ) are indecomposable. Let $\phi_1,\ldots,\phi_d : U \to V$ be the linear bijections that define $(U,\Delta) \simeq_s (V,\Theta)$. For each i, we can write $\phi_i = a_i\phi_1$ for some $a_i \in Z(V,\Theta)^\times$ by Proposition 3.3. As Θ is indecomposable, the center $Z(V,\Theta)$ is local and its residue field is $\mathbb{R}$ or $\mathbb{C}$. If the residue field is $\mathbb{C}$, then each a_i is a d-th power in $Z(V,\Theta)$ by Lemma 3.4. The same argument as Theorem 1.4 (3) shows that $\Delta \cong \Theta$. If the residue field is $\mathbb{R}$, then a_i or $-a_i$ is a d-th power, and we have $\Delta \cong \Theta$ or $\Delta \cong -\Theta$ in the same manner. In general, by (1) of Theorem 1.4 we have $(U,\Delta) \cong (U_0,0)\oplus(U_1,\Delta_1')\oplus\cdots\oplus(U_r,\Delta_r')$ and $(V,\Theta) \cong (V_0,0)\oplus(V_1,\Theta_1')\oplus\cdots\oplus(V_r,\Theta_r')$ such that all Δ_i''s and Θ_i''s are indecomposable and $\Delta_i' \simeq_s \Theta_i'$ for each i. By the previous argument on indecomposable forms, we have $\Delta_i' \cong \Theta_i'$ or $\Delta_i' \cong -\Theta_i'$. Let Δ_1 be the direct sum of Δ_0 and all Δ_i''s such that $\Delta_i \cong \Theta_i'$, and let Δ_2 be the direct sum of the rest Δ_i''s. Similarly define Θ_1 and Θ_2. Then we have $\Delta = \Delta_1 \oplus \Delta_2$ and $\Theta = \Theta_1 \oplus \Theta_2$ with $\Delta_1 \cong \Theta_1$ and $\Delta_2 \cong -\Theta_2$. □

Example 3.7. Let $A = \mathbb{R}^2$ be the algebra with product $(x_1,y_1)(x_2,y_2) = (x_1x_2, y_1y_2)$. Choose two $\mathbb{R}$-linear forms l_1 and l_2 of A as: $l_1(x,y) = x+y$ and $l_2(x,y) = x-y$. Then we have two associated 4-linear forms on A as Example 2.6: $\Theta(a_1,a_2,a_3,a_4) = l_1(a_1a_2a_3a_4)$ and $\Delta(a_1,a_2,a_3,a_4) = l_2(a_1a_2a_3a_4)$ for all $a_1,a_2,a_3,a_4 \in A$. Let ϕ be the $\mathbb{R}$-linear bijection of A such that $\phi(x,y) = (x,-y)$. Then Θ and Δ are symmetrically equivalent with respect to the bijections $\{\phi, \mathrm{Id}_A, \mathrm{Id}_A, \mathrm{Id}_A\}$, where Id_A is the identity map on A. However, they are not isomorphic to each other since Θ is nonnegative on $\{(a,a,a,a) : a \in A\}$ but Δ may take negative values. It is easy to see that there exist the direct sum decompositions $\Theta = \Theta_1 \oplus \Theta_2$ and $\Delta = \Theta_1 \oplus -\Theta_2$, where Θ_1 and Θ_2 are two 4-linear forms on the subspaces $V_1 = \mathbb{R}(1,0)$ and $V_2 = \mathbb{R}(0,1)$ respectively such that

$$\Theta_1\big((x_1,0),(x_2,0),(x_3,0),(x_4,0)\big) = x_1x_2x_3x_4,$$
$$\Theta_2\big((0,y_1),(0,y_2),(0,y_3),(0,y_4)\big) = y_1y_2y_3y_4.$$

4. Recovery of homogeneous polynomials from their Jacobian ideals

In this section, the ground field $\Bbbk$ is assumed to be algebraically closed and of characteristic 0 or greater than d. We apply Theorem 1.4 to provide a linear algebraic proof for Theorem 1.5, the well known Torelli type result of Donagi [6, Proposition 1.1]. Let $f(x_1,\dots,x_n) \in \Bbbk[x_1,\dots,x_n]$ be a homogeneous polynomial of degree d. Let $J(f)$ be its Jacobian ideal generated by $\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}$. The classical Torelli problem concerns about how to recover f from $J(f)$.

Homogeneous polynomials are naturally associated to symmetric multilinear forms and symmetric tensors, see e.g. [10,11]. Write f in the symmetric way

$$f(x_1,\dots,x_n) = \sum_{1\le i_1,\dots,i_d\le n} a_{i_1\cdots i_d} x_{i_1}\cdots x_{i_d},$$

where the $a_{i_1\cdots i_d}$'s are symmetric with respect to their indices. Let V be an n-dimensional $\Bbbk$-space with a basis $e_1,\dots,e_n$. Define the symmetric d-linear form $\Theta\colon V\times\cdots\times V \longrightarrow \Bbbk$ by

$$\Theta(e_{i_1},\dots,e_{i_d}) = a_{i_1\cdots i_d}, \quad 1 \le i_1,\dots,i_d \le n.$$

The pair (V, Θ) is called the associated symmetric d-linear form of f under the basis $e_1, \ldots, e_n$. The homogeneous polynomial f and the symmetric multilinear form (V, Θ) is explicitly related as

$$f(x_1, \ldots, x_n) = \Theta\Big(\sum_{1 \le i \le n} x_i e_i,\ \ldots,\ \sum_{1 \le i \le n} x_i e_i\Big).$$

For each $1 \le i \le n$, let Θ_i be the $(d-1)$-linear form on V such that

$$\Theta_i(v_1, \ldots, v_{d-1}) = \Theta(e_i, v_1, \ldots, v_{d-1}), \quad \text{for all } v_1, \ldots, v_{d-1} \in V.$$

It is well known that

$$\frac{1}{d}\frac{\partial f}{\partial x_i} = \Theta_i\Big(\sum_{1 \le j \le n} x_j e_j,\ \ldots,\ \sum_{1 \le j \le n} x_j e_j\Big). \tag{4.1}$$

In other words, Θ_i is associated to $\frac{1}{d}\frac{\partial f}{\partial x_i}$ under the basis $e_1, \ldots, e_n$. Recall that centers of symmetric multilinear forms can also be equivalently defined in terms of homogeneous polynomials, see [9,10]. Let H be the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{1 \le i,\ j \le n}$ of f and define its center $Z(f)$ as

$$Z(f) = \{X \in \mathbb{k}^{n \times n} \mid (HX)^T = HX\}. \tag{4.2}$$

It is clear that $Z(V, \Theta) \cong Z(f)$.

Example 4.1. Consider the homogeneous polynomial $f(x, y, z) = x^3 + y^2 z$. By (4.2), we can easily compute that

$$Z(f) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & c & b \end{pmatrix} \middle| a, b, c \in \mathbb{k} \right\}.$$

Note that $Z(f)$ has nontrivial idempotents and is not semi-simple. It follows that f is decomposable and singular, see [11] for more details.

In [5], Carlson and Griffiths proved a similar result of Theorem 1.5: If f is required to be generic, then $J(f) = J(g)$ implies that $g = \lambda f$ for some $\lambda \in \mathbb{C}$. The authors showed in [11, Theorem 3.11] that a higher degree form f is determined by its Jacobian ideal $J(f)$, up to a nonzero constant factor, if and only if $Z(f) \cong \mathbb{k}$. The proof is completely linear algebraic, in which the center theory is the key. Here by combining the theories of centers and symmetric equivalence, we are able to provide a linear algebraic proof for Theorem 1.5. Moreover, our proof is constructive, in contrast to the proofs in the literature which only guarantee the existence. Two examples are included to elucidate the present approach after the proof of Donagi's theorem.

Proof of Theorem 1.5. Let Θ and Δ be the d-linear forms associated to f and g respectively. Let $E(f)$ be the vector space spanned by $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ which is the $(d-1)$-th homogeneous part of $J(f)$. If $\dim E(f) < n$, then f is degenerate and let V_0 be the subspace consisting of the vectors along which the partial derivative of f is zero and choose another subspace V_1 such that $V = V_0 \oplus V_1$. Then we have a decomposition $(V, \Theta) = (V_0, 0) \oplus (V_1, \Theta|_{V_1})$ by Theorem 1.4. As $J(f) = J(g)$ and thanks to Equation (4.1), we have a similar direct sum decomposition $(V, \Delta) = (V_0, 0) \oplus (V_1, \Delta|_{V_1})$. Moreover the restrictions of f and g on V_1 also have the same Jacobian ideal. Therefore, we reduce the theorem to the nondegenerate case.

As $J(f) = J(g)$ and $\dim E(f) = n$, there exists a matrix $A = \left(a_{ij}\right)_{n\times n} \in GL_n(\Bbbk)$ such that

$$\Big(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\Big) = \Big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\Big)A. \tag{4.3}$$

Define $\phi \in \mathrm{End}(V)$ by $\phi(e_i) = \sum\limits_{j=1}^{n} a_{ji}e_j$ for all $1 \le i \le n$. Note by [11, Lemma 3.10] that $A \in Z(f)$, hence $\phi \in Z(V, \Theta)$. In addition, for all $1 \le i_1, \ldots, i_d \le n$ we have

$$\begin{aligned}
\Theta(\phi(e_{i_1}), e_{i_2}, \ldots, e_{i_d}) &= \Theta\Big(\sum_{j=1}^{n} a_{ji_1}e_j, e_{i_2}, \ldots, e_{i_d}\Big) \\
&= \sum_{j=1}^{n} a_{ji_1}\Theta_j(e_{i_2}, \ldots, e_{i_d}) \\
&= \Delta_{i_1}(e_{i_2}, \ldots, e_{i_d}) \quad (\textit{by } (4.1)\ \&\ (4.3)) \\
&= \Delta(e_{i_1}, e_{i_2}, \ldots, e_{i_d}).
\end{aligned}$$

It follows immediately that $\phi, \mathrm{Id}, \ldots, \mathrm{Id}$ make a symmetric equivalence between Θ and Δ. Then by item (3) of Theorem 1.4, f and g are equivalent up to an invertible linear transformation. $\square$

Example 4.2. Take (V, Θ) as the 3-linear form in Example 2.3. Consider another 3-linear form (V, Δ) with $\Delta((x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)) = x_1y_1z_1 + 8x_2y_2z_2 - x_3y_3z_3$. Note that Θ and Δ are both symmetric and their associated homogeneous polynomials are $f(x, y, z) = x^3 + y^3 + z^3$ and $g(x, y, z) = x^3 + 8y^3 - z^3$ respectively. It is easy to see that

$$\Big(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\Big) = \Big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\Big)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence f and g have the same Jacobian ideal. Then according to Theorem 1.4 and its proof, one can easily observe that $g(x, y, z) = f(x, 2y, -z)$.

Example 4.3. Let $f(x,y,z) = 2x^3 + 4x^2y + 3xy^2 + y^3 + 3x^2z + 2xyz + 3xz^2 + y^2z + z^3$ and $g(x,y,z) = 2x^3 - 2x^2y - 3xy^2 - y^3 + 9x^2z + 2xyz + 9xz^2 + yz^2 + 3z^3$. Then we have

$$\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \begin{pmatrix} -7 & -2 & -6 \\ 6 & 1 & 6 \\ 8 & 2 & 7 \end{pmatrix}.$$

So f and g have the same Jacobian ideal. It follows by Theorem 1.5 that f and g are equivalent up to a change of variables. We compute the center

$$Z(f) = \left\{ a \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} + c \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \Bbbk \right\},$$

and find that the matrix $A = \begin{pmatrix} -7 & -2 & -6 \\ 6 & 1 & 6 \\ 8 & 2 & 7 \end{pmatrix}$ belongs to $Z(f)$ by the following equality

$$\begin{pmatrix} -7 & -2 & -6 \\ 6 & 1 & 6 \\ 8 & 2 & 7 \end{pmatrix} = - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} + 6 \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Take a cubic root $B = \begin{pmatrix} -3 & -2 & -2 \\ 2 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ of A in $Z(f)$ as what we have done in the proof of Theorem 1.4. Then g is equivalent to f under the invertible linear transformation which sends $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to $B \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. In other words,

$$g(x,y,z) = f(-3x - 2y - 2z, 2x + y + 2z, 4x + 2y + 3z).$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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